ON THE ESTIMATE FOR THE AUTOCOVARIANCE FUNCTION OF HOMOGENEOUS AND ISOTROPIC VECTOR-VALUED RANDOM FIELD ON THE SPHERE

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Abstract

In this paper we study asymptotic distribution estimate of the autocovariance function $\widehat{B}^{(T)}(\theta,t)$ of homogeneous and isotropic vector-valued random fields $\vec{\xi}(P,t)$.

Different problems of the theory of automatic control, radio physics, geochemistry, the meteorologies result in necessity to consider stochastic functions time-dependent and a point on an sphere. For want of it it appears natural the suppositions about a homogeneous in time variable and isotropic on a space variable of random fields. The users of many statistical application software packages apply to an estimation of spectral characteristic of random processes and fields of statistics such as a smoothed periodogram, statistics such as average periodograms, that is used thes paper.

Let $\vec{\xi}(P,t), t \in S_n$ a homogeneous on time variable and isotropic on a space variable column vector-valued random field with components $\xi_a(P,t), a = 1,\dots,r$ on $S_n \times Z$, where $S_n -$ is the unit sphere of the n-rd dimensional space R^n and $Z = 0,\pm 1,\dots$, which has finite second order moment and which for each t is continuous in quadratic mean (q.m.) . $E\vec{\xi}\ \ (P,t) = \mu, \mu -$ unknown.

$E[\vec{\xi}(P,t+s)-\mu][\vec{\xi}(Q,s)-\mu] = B(\theta,t)$, where θ- is the angular distance between Р и Q.

It is known that $\vec{\xi}(P,t)$ can be represented as (see Yadrenko (!983),) Korolyuk(1985))

$$\vec{\xi}(P,t) = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \vec{\xi}_m^l(t) S_m^l(\mathrm{P}),$$

where the $S_m^l(\mathrm{P})$ are the orthonormal spherical harmonics of degree m,

$$h(m,n) = (2m+n-2)\frac{(m+n-3)!}{(n-2)!m!}\ -$$

is the number of linearly independent spherical harmonics of degree m,

$$E\,{}_a\xi_m^l(t)\,{}_b\xi_{m_1}^{l_1}(s) = \delta_m^{m_1}\delta_l^{l_1}\int_{-\pi}^{\pi} e^{i\lambda(t-s)} d_{a,b}F_m(\lambda),$$

and $\{\vec{F}_m(\lambda), m \ge 0\}$ is a sequence of real nondecreasing function such that

$$\sum_{m=0}^{\infty} h(m,n)\int_{-\pi}^{\pi} d_{i,i}F_m(+\infty) < +\infty, i = 1,\dots,r$$

δ_k^l being the Kronecker delta.

The autocovariance matrix function $B(\theta,t)$ can be written as

$$\vec{B}(\theta,t)=\omega_n^{-1}\sum_{m=0}^{\infty}h(m,n)\frac{C_m^{\frac{n-2}{2}}(\cos\theta)}{C_m^{\frac{n-2}{2}}(1)}\int_{-\pi}^{+\pi}e^{it\lambda}d\vec{F}_m(\lambda),$$

where $\omega_n=2\pi^{n/2}/\Gamma(n/2)$ is the area of the surface S_n and the $C_m^{\upsilon}(x),(\upsilon\neq 0)$ are the Gegenbauer polynomials.(see Beteman H.,Erdelyi A.(1953))

We shall assume that $\vec{\xi}(P,t)$ is stationary in time and homogeneous and isotropic in space in the strict sense, i.e. for any finite sequence of points $(P_1,t_1),\ldots,(P_k,t_k)$ the distribution of $\vec{\xi}(g(P_1),t_1+t),\cdots\vec{\xi}(g(P_k),t_k+t)$ does not depend on g and t for all $g\in G$ and $t_k, k=1,2,\ldots$ where G is the group of rotations of R^n about the origin.

The following conditions let are executed:

For each t $\{\vec{\xi}(P,t): P\in S_n\}$ is continuous in q.m.

Let $\left|E\vec{\xi}(P_1,t_1)\cdots\vec{\xi}(P_k,t_k)\right|\leq m_k(t_1,\ldots,t_k)$ uniformly in $P_1,\ldots,P_k$ for all $t_1,\ldots,t_k; k=1,2,\ldots$.

For $(x_1,x_2,\ldots x_k)$ we denote its joint cumulant of order k by $cum(x_1,\ldots,x_k)$.

Condition B. For a given $q\geq 0$

$$\sum_{u_1,\ldots,u_{k-1}=-\infty}^{\infty}\{1+|u_j|^q|cum\{\vec{\xi}(P_1,t+u_1),\ldots,\vec{\xi}(P_{k-1},t+u_{k-1}),\vec{\xi}(P_k,t)\}|du_1\ldots du_{k-1}\leq C_k<\infty$$

uniformly in $P_1,P_2,\ldots,P_k$ for $j=1,2,\ldots,k-1$ and $k=2,3,\ldots$.

This condition is analogous to the one made by Brillinger (1969,1970).

Let field $\vec{\xi}(P,t)$ is observed for all points of S_n and on time $t=0,1,\ldots,T-1$.

For any ℓ we define

$$_a d_m^{(T)}(\lambda)=\sum_{t=0}^{T-1}{}_a\xi_m^l(t)e^{-i\lambda t}$$

The periodogram

$$_{ab}I_m^{(T)}(\alpha)=(2\pi T)^{-1}d_a^{(T)}(\alpha)d_b^{*(T)}(-\alpha)$$

is an asymptotically unbiased estimator of the spectral density $_{a,b}f_m(\lambda)$ The * - denotes line-vector. This estimate is not consistent. In order to construct a consistent estimator we consider a bounded even function $H(\alpha),-\pi\leq\alpha\leq\pi$, that has a bounded derivative and such that

$$\int_{-\pi}^{\pi}H(\alpha)d\alpha=1\cdot$$

For a sequence of positive numbers A_T we set $H^{(T)}(\alpha)=A_T^{-1}H(A_T^{-1}\alpha)$.We define an estimator for the spectral density in the following manner:

$$_{a,b}\widehat{f}_m^{(T)}(\lambda)=\int_{-\pi}^{+\pi}H^{(T)}(\alpha)\,_{a,b}I_m^{(T)}(\lambda-\alpha)d\alpha$$

As an estimate of the autocovariance function of a random field the following statistics is considered:

$$\widehat{B}^{(T)}(\theta,t)=\omega_n^{-1}\sum_{m=0}^{N_T}h(m,n)\frac{C_m^{\frac{n-2}{2}}(\cos\theta)}{C_m^{\frac{n-2}{2}}(1)}\widehat{r}_m^T(t)$$

where $_{a,b}\widehat{r}_m^T(t)=\int_{-\pi}^{\pi}e^{i\alpha t}\,_{a,b}\widehat{f}_m^{(T)}(\alpha)d\alpha$, for each T, N_T is a positive integer and $N_T\to\infty$ as $T\to\infty,\theta\in[0,\pi]$.

We assume that second-order cumulant spectra of $\vec{\xi}(P,t)$ satisfy

$$\sum_{m=0}^{\infty} h^2(m,n) \int_0^{2\pi} \vec{f}_m^{\,2}(\alpha) d\alpha < \infty$$

and the fourth-order cumulant spectra are such that

$$\sum_{m,q=0}^{\infty} h(m,n)h(q,n) \int_0^{2\pi} \int_0^{2\pi} \vec{g}_{m,q}(\alpha,-\alpha,\beta) d\alpha d\beta < \infty .$$

Theorem: Under the above assumptions the estimate $\widehat{B}^{(T)}(\theta,t)$ is asymptotically unbiased and asymptotically consistent.

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